

COMPLETIONS OF INFINITESIMAL HECKE ALGEBRAS OF \mathfrak{sl}_2

AKAKI TIKARADZE

ABSTRACT. We relate completions of infinitesimal Hecke algebras of \mathfrak{sl}_2 to noncommutative deformations of Kleinian singularities of type D of Crawley-Boevey and Holland. As a consequence, we show an analogue of Bernstein's inequality and simplicity of generic maximal primitive quotients of these algebras. We also establish Skryabin type equivalence for these algebras.

1. INTRODUCTION

Given an associative Noetherian \mathbb{C} -algebra A of finite Gelfand-Kirillov dimension, it is natural to ask if (generalized) Bernstein's inequality holds: Is it true that for any finitely generated A -module M one has $GK_A(M) \geq \frac{1}{2}GK(A/\text{Ann}(M))$? (here GK stands for Gelfand-Kirillov dimension). We show that this is the case for infinitesimal Hecke algebras of \mathfrak{sl}_2 .

Recall that for a reductive Lie algebra \mathfrak{g} and its finite dimensional representation V , Etingof, Gan and Ginzburg [EGG] defined a family of PBW deformations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g} \ltimes V)$ of the semi-direct product Lie algebra $\mathfrak{g} \ltimes V$, which they called infinitesimal Hecke algebras.

There is a particularly nice family of infinitesimal Hecke algebras for the pair $\mathfrak{g} = \mathfrak{sl}_2, V = \mathbb{C}^2$. These algebras, denoted by H_z , depend on a parameter z which is an element of the center of $\mathcal{U}\mathfrak{sl}_2$. Algebras H_z were studied by Khare and the author [Kh, KT]. In this paper, we relate noncommutative deformations of Kleinian singularities of type D of Crawley-Boevey and Holland (which are spherical subalgebras of symplectic reflection algebras for $\dim V = 2$) [CBH] to infinitesimal Hecke algebras of \mathfrak{sl}_2 . Namely, we show that a certain completion of the central quotient of an infinitesimal Hecke algebra of \mathfrak{sl}_2 is isomorphic to the tensor product of the completed Weyl algebra in two generators and the completion of an algebra of Crawley-Boevey and Holland. As a consequence, we establish Bernstein's inequality for these algebras (Theorem 4.1.) We also show an equivalence between certain subcategory of modules over H_z and the category of modules over corresponding noncommutative deformations of type D singularities (Theorem 5.1).

2. COMPLETIONS OF ALMOST COMMUTATIVE ALGEBRAS

We will always work over the field of complex numbers \mathbb{C} . In this section we will recall some necessary constructions and fix notations related to a slice algebra construction due to Losev [L1].

Throughout we will use the following convention: For a commutative algebra B and a closed point $y \in \operatorname{Spec} B$, $B_{\bar{y}}$ will denote the completion of B with respect to the maximal ideal m_y corresponding to y . Also, for a symplectic variety Y and $y \in Y$ we will denote by $W_{\hbar}(Y_{\bar{y}})$ the completed Weyl algebra, a deformation quantization of $\mathcal{O}(Y)_{\bar{y}}$, where \hbar is the deformation parameter.

By an almost commutative algebra we will mean an associative \mathbb{C} -algebra equipped with an ascending filtration

$$\mathbb{C} = A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots, \quad \bigcup_{n \in \mathbb{N}} A_n = A, \quad A_n A_m \subset A_{n+m},$$

such that the associated graded algebra is a finitely generated commutative ring over \mathbb{C} . Recall that in this case $\operatorname{gr} A$ comes equipped with a natural Poisson bracket. The origin of $\operatorname{Spec} \operatorname{gr} A$ will be denoted by $\{0\}$.

By a generic subset of an algebraic variety we will mean Weil generic subset, i.e. a set which is the complement of a countable union of proper closed subvarieties.

Let A be an almost commutative algebra equipped with a filtration A_n ($n \geq 0$) over \mathbb{C} . Let $Y \subset \operatorname{Spec} \operatorname{gr} A = X$ be an algebraic symplectic leaf $\dim Y = 2d$, and let $y \in Y$. We will recall the slice algebra construction of Losev [L1]. One starts by completing the Rees algebra $R(A) = \bigoplus_n A_n \hbar^n \subset A[\hbar]$ with respect to the ideal $p^{-1}(m_y)$, where $m_y \subset \operatorname{gr} A$ is the maximal ideal corresponding to y , and $p : R(A) \rightarrow R(A)/\hbar = \operatorname{gr} A$ is the natural projection. This completed algebra will be denoted by $R(A)_{\bar{y}}$. Then according to Losev, $R(A)_{\bar{y}}$ is a free $\mathbb{C}[[\hbar]]$ -module and $R(A)_{\bar{y}}/\hbar = (\operatorname{gr} A)_{\bar{y}}$, the completion of $\operatorname{gr} A$ at m_y .

According to Kaledin [?], $(\operatorname{gr} A)_{\bar{y}}$ is isomorphic to the completed tensor product $\operatorname{Spec} Y_{\bar{y}} \otimes B$, where B is a complete Poisson algebra with the origin being a symplectic leaf of $\operatorname{Spec} B$. As proved by Losev [L1], this decomposition can be lifted to $R(A)_{\bar{y}}$, meaning that there is a free $\mathbb{C}[[\hbar]]$ -subalgebra A_y^\spadesuit , a *slice algebra*, such that $R(A)_{\bar{y}}$ is identified with the completed tensor product $W_{\hbar}(Y_{\bar{y}}) \otimes_{\mathbb{C}[[\hbar]]} A_y^\spadesuit$, such that $A_y^\spadesuit/\hbar = B$.

If M is a finitely generated left A -module, then M can be equipped with a filtration compatible with the filtration of A such that the corresponding Rees module $R(M)$ is finitely generated over $R(A)$ (a good filtration). As before, one defines $R(M)_{\bar{y}}$ as the completion of $R(M)$ with respect to $p^{-1}(m_y)$, so $R(M)_{\bar{y}} = R(M) \otimes_{R(A)} R(A)_{\bar{y}}$ which is a nonzero finitely generated $R(A)_{\bar{y}}$ -module when $y \in \operatorname{Supp}(\operatorname{gr} M)$. In which case $R(M)_{\bar{y}}/\hbar = (\operatorname{gr} M) \otimes_{\operatorname{gr} A} (\operatorname{gr} A)_{\bar{y}} = (\operatorname{gr} M)_{\bar{y}}$. We will denote by $SS(M)$

the corresponding characteristic variety $\text{Supp}(\text{gr } M) \subset \text{Spec } \text{gr } A$. Recall the following observation of Losev.

Proposition 2.1. *Let $I \subset A$ be a two-sided ideal of A such that \bar{Y} is a connected component of $V(\text{gr } I)$, where as before Y is a symplectic leaf in $\text{Spec } \text{gr } A$. Then for any $y \in Y$, there exists a nonzero left A_y^\bullet -module which is a finitely generated free module over $\mathbb{C}[[\hbar]]$.*

We recall the proof for the convenience of the reader.

Proof. We have that $\bar{I} = R(I)R(A)_{\bar{y}}$ is a two-sided ideal of $R(A)_{\bar{y}}$, such that $R(A)_{\bar{y}}/\bar{I}$ has support $Y_{\bar{y}}$. Therefore, any finitely generated A_y^\bullet -submodule of $R(A)_{\bar{y}}/\bar{I}$ is supported at the origin in $\text{Spec } A_y^\bullet/\hbar$, hence is finite-dimensional. \square

For a finitely generated module M over an almost commutative algebra A , we will denote its Gelfand-Kirillov dimension by $GK(M)$, as usual. Recall that in the above setting $GK(M) = \dim SS(M)$.

3. COMPLETIONS OF INFINITESIMAL HECKE ALGEBRAS OF \mathfrak{sl}_2

We will denote by Δ the rescaled Casimir element $\Delta = h^2 + 4fe + 2h \in \mathfrak{U}\mathfrak{sl}_2$, where h, e, f denote the standard basis elements for \mathfrak{sl}_2 . For a given $z' \in \mathbb{C}[\Delta]$, the algebra $H_{z'}$ is the quotient $\mathfrak{U}\mathfrak{g} \ltimes TV/([x, y] - z')$, where $\mathfrak{g} = \mathfrak{sl}_2$ and $V = \mathbb{C}x \oplus \mathbb{C}y$ is its standard 2-dimensional representation with relations

$$[e, x] = 0, \quad [f, x] = y, \quad [h, x] = x.$$

It was shown in [KT] that center of $H_{z'}$ is generated by $t_z = ey^2 + \frac{1}{2}h(xy + yx) - fx^2 + z$, where $z \in \mathbb{C}[\Delta]$ is an element uniquely defined by z' up to adding a constant (but z' is uniquely determined by z). One has $\deg(z) = \deg(z') + 1$, and the leading coefficient of z as a polynomial in Δ is $\frac{1}{2\deg(z')+1}$ times the leading coefficient of z' . Let U_z be the quotient $H_{z'}/(t_z)$. We will introduce the following filtration on U_z :

$$\deg e = \deg f = \deg h = 2, \quad \deg x = \deg y = 2n + 1,$$

where $n = \deg z'$ as a polynomial in Δ . From now on we will assume that $n \geq 2$ and z is monic in Δ . In what follows for a filtered algebra A , given an element $a \in A$, we will denote for simplicity $\text{gr } a \in \text{gr } A$ still by a whenever it will not cause a confusion. It follows easily from PBW property that $\text{gr } U_z$ is the quotient of the polynomial algebra $\mathbb{C}[e, f, h, x, y]$ by the principal ideal $(ey^2 + hxy - fx^2 - (\Delta)^{n+1})$, where $\Delta = h^2 + 4ef$. We will denote this Poisson algebra by B_n . Thus, the Poisson bracket on B_n is defined as follows

$$\begin{aligned} \{e, x\} &= 0 = \{f, y\}, \{f, x\} = y, \{x, y\} = (2n + 1)\Delta^n, \\ \{h, e\} &= 2e, \{h, f\} = -2f, \{e, f\} = f, \{e, y\} = x. \end{aligned}$$

We note that $SL_2(\mathbb{C})$ acts naturally on $H_{z'}, U_z$ preserving the corresponding filtration. This action gives rise to a natural action of $SL_2(\mathbb{C})$ on B_n preserving the Poisson bracket.

We now describe the symplectic leaves of $\text{Spec } B_n$.

Proposition 3.1. *B_n is a normal integral domain. The singular locus of $\text{Spec } B_n$ is $\{x = y = 0\}$, and its smooth locus is symplectic. The symplectic leaves of $\text{Spec } B_n$ are the origin $\{0\}$, $V(I) \setminus \{0\}$, and its smooth locus $\text{Spec } B_n \setminus \{x = y = 0\}$.¹*

Proof. It is easy to see that B_n is a domain. It is clear that the ideal $I = (x, y, \Delta)$ is a Poisson ideal and $V(I)$ (the zero locus of I) belongs to the singular locus of $\text{Spec } B_n$. Let us take $a \in \text{Spec } B_n \setminus V(I)$. Let m_a be the corresponding maximal ideal. The m_a -adic completion of B_n will be denoted by $B_{\bar{a}}$ for brevity. We will show that $\text{Spec } B_{\bar{a}}$ is a (formal) symplectic variety. Using the action of $SL_2(\mathbb{C})$ on $\text{Spec } B_n$ we may assume without loss of generality that $e(a) \neq 0$. It is easily seen that $B_{\bar{a}}$ is isomorphic to the tensor product of $\mathbb{C}[[e^{-1}h - \frac{h(a)}{e(a)}, e - e(a)]]$ and of the completion of $\mathbb{C}[A, C, \Delta]/(C^2 - \Delta(\frac{1}{4}A^2 + \Delta^n))$ at the point $(A(a), C(a), \Delta(a))$, where $A = e^{-\frac{1}{2}}x$, $C = e^{\frac{1}{2}}y + \frac{1}{2}hA$. The latter is the ring of functions of the Kleinian singularity of type D_{n+2} , which is symplectic outside the origin. Thus, $\text{Spec } B_n \setminus V(I)$ is a symplectic. On the other hand, $V(I)$ considered as a reduced subvariety of $\text{Spec } B_n$ is the nilpotent cone of \mathfrak{sl}_2 . Hence $V(I) \setminus \{0\}$ is a symplectic leaf of $\text{Spec } B_n$. Normality of B_n follows Serre's criterion, since B_n is regular in codimension 1. \square

The following is an analogue of Kostant's theorem.

Proposition 3.2. *The algebra H_z is a free module over its center.*

Proof. It suffices to check that $\text{gr } H_z$ is a free module over $\mathbb{C}[\text{gr } t_z]$. Let us introduce another filtration of $\text{gr } H_z$ by setting $\deg x = \deg y = 0$, $\deg e = \deg f = \deg h = 1$. Then under the new filtration, $\text{gr}(\text{gr } t_z) = \Delta^{n+1}$. But by Kostant's theorem, $\text{Sym } \mathfrak{g}$ is free over $\mathbb{C}[\Delta^{n+1}]$. Thus we conclude that $\text{gr}(\text{gr}(H_z)) = \mathbb{C}[x, y] \otimes \text{Sym } \mathfrak{g}$ is free over $\text{gr}(\text{gr } t_z)$. Hence $\text{gr } H_z$ is free over $\mathbb{C}[\text{gr } t_z]$. \square

Recall that for a given finite group $\Gamma \in SL_2(\mathbb{C})$, and a central element of the group algebra $\lambda \in \mathbb{C}[\Gamma]$, Crawley-Boevey and Holland [CBH] defined an algebra \mathcal{O}^λ as $e(T(V)/([x, y] - \lambda))e$, where $e = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g$ and x, y is the standard basis of $V = \mathbb{C}^2$.

In what follows we will consider square roots of certain non-central elements. Let us clarify this and fix the appropriate notation. Let A be an associative \mathbb{C} -domain, and let $e \in A$ be an element such that $\text{ad}(e)$ acts locally nilpotently on A . Moreover, assume that A is separated in $(e - 1)$ -adic topology, i.e. $\cap_n (e - 1)^n = 0$. Then we will denote by $A[e^{-\frac{1}{2}}]$ the subalgebra of the completion of A by $(e - 1)$ generated over A by $e^{-\frac{1}{2}} = \exp(-\frac{1}{2} \log e)$, where $\log e$ is understood as a power series in $(e - 1)$.

¹Apoorva Khare has obtained this answer earlier using a different computation.

Also, by $W_1(R)$ we will denote the Weyl algebra over R with 2 generators: $W_1(R) = R\langle x, y \rangle / ([x, y] - 1)$.

The following result was motivated by Losev's work on completions of symplectic reflection algebras [L2].

Theorem 3.1. *For any $a \in Y = V(I) \setminus \{0\}$, the algebra $R(U_z)_{\bar{a}}$ is isomorphic to the completed tensor product $W_{\hbar}(Y_{\bar{a}}) \otimes_{\mathbb{C}[[\hbar]]} R(\mathcal{O}_{\lambda(z)})_{\bar{0}}$, where $\mathcal{O}^{\lambda(z)}$ is the noncommutative deformation of the Kleinian singularity of type D_{n+2} (parameter $\lambda(z)$ will be determined in the proof). For generic z , the algebra U_z is simple.*

Proof. Since $SL_2(\mathbb{C})$ acts transitively on $V(I) \setminus \{0\}$, we may assume without loss of generality that a is the point with coordinates $e = 1, h = f = 0$. It is straightforward to check that the algebra $U_z[e^{-\frac{1}{2}}]$ is generated by $e^{\frac{1}{2}}, e^{-\frac{1}{2}}$ over U_z subject to the following relations:

$$\begin{aligned} [f, e^m] &= -mhe^{m-1} + m(m-1)e^{m-1}, & [y, e^m] &= -me^{m-1}xe^{m-1}, \\ [h, e^m] &= 2me^m, & [x, e^m] &= 0 \end{aligned}$$

for $m \in \frac{1}{2}\mathbb{Z}$. We have that $[\frac{1}{2}e^{-1}h, e-1] = 1$, and both $\frac{1}{2}e^{-1}h, e-1$ commute with $A = e^{-\frac{1}{2}}x$, $C = e^{\frac{1}{2}}y + \frac{1}{2}hA$, $\Delta = h^2 + 4fe + 2h$. Let us denote the subalgebra of $U_z[e^{-\frac{1}{2}}]$ generated by A, C, Δ by U'_z . It is easy to see that $U_z[e^{-\frac{1}{2}}] = \mathbb{C}[\frac{1}{2}e^{-1}h, e-1][e^{-\frac{1}{2}}] \otimes U'_z$. Direct computation shows that the following relations hold in U'_z :

$$\begin{aligned} [\Delta, C] &= \Delta A + (A - C), & [A, C] &= z' - \frac{1}{2}A^2, \\ [\Delta, A] &= 4C - A, & z + \frac{1}{4}\Delta A^2 - \frac{1}{2}AC &= C^2. \end{aligned}$$

Now we will need to recall the explicit relations for \mathcal{O}^λ (noncommutative deformations of type D_{n+2} singularities). Recall that Levy [Le] has defined the following algebras $D(Q, \gamma)$ for a polynomial $Q(t)$, and $\gamma \in \mathbb{C}$, with generators u, v, w , and relations:

$$[u, v] = 2w, \quad [u, w] = 2uv + 2w + \gamma, \quad [v, w] = v^2 + P(u)$$

and

$$Q(u) + uv^2 + w^2 2wv\gamma v = 0,$$

where P is the unique polynomial such that

$$Q(-s(s-1)) - Q(-s(s+1)) = (s-1)P(-s(s-1)) + (s+1)P(-s(s+1)).$$

Similarly, Boddington has defined an algebra $D(q)$ depending on a polynomial q , and has showed that $D(q)$ is isomorphic to $D(Q, \gamma)$ when $\gamma = 2q(\frac{1}{2})$ and $Q(-u + \frac{-1}{4}) = [-\sqrt{u} - \frac{1}{2}p(\sqrt{u})]$, where $p(x) = \frac{-4q(x)q(-x-1) + \gamma^2}{(1+2x)^2}$ and notation $[f(\sqrt{x})] \in \mathbb{C}[x]$ for a polynomial $f(x)$ means the following:

$f(\sqrt{x}) = h(x) + \sqrt{x}[f(\sqrt{x})]$ for unique polynomials $h(x), [f(\sqrt{x})]$. Bodington [Bo] shows that $D(q)$ is isomorphic to \mathcal{O}^λ where λ is the tuple $(\lambda_a, \lambda_b, \lambda_1, \dots, \lambda_{n-1}, \lambda_c, \lambda_d)$ such that $q(x) = \prod_{i=0}^{n-1} (x + \mu_i)$, where

$$\mu_0 = \frac{1}{2}\lambda_a - \lambda_b, \mu_1 = \frac{1}{2}(\lambda_a + \lambda_b), \mu_2 = \mu_1 + \lambda_1, \dots, \mu_{n-1} = \mu_1 + \lambda_1 + \dots + \lambda_{n-1} + \lambda_c$$

Direct computation shows that our algebra U'_z is isomorphic to $D(Q, 0)$ for $Q(u) = 3z'(-u - \frac{3}{4}) - z(-u - \frac{3}{4})$. This can be seen by putting first $\Delta = -u, C = \frac{1}{2}w, A = -v$, and then replacing u, w by $u + \frac{3}{4}$ and $w + \frac{1}{2}$ respectively. (We denote the corresponding parameter by $\lambda(z)$.) Then we see that for generic z , $\lambda(z)$ is the generic parameter such that $q(\frac{1}{2}) = 0$, i.e. $\mu_i = -\frac{1}{2}$ for some i . Thus, U'_z is isomorphic to $\mathcal{O}^{\lambda(z)}$, and for generic z , $\mathcal{O}^{\lambda(z)}$ is simple.

To summarize, $U_z[e^{-\frac{1}{2}}] \cong W_1(\mathbb{C})[e^{-\frac{1}{2}}] \otimes \mathcal{O}^{\lambda(z)}$, where the Weyl algebra $W_1(\mathbb{C})$ is defined by the following generators and relations: $W_1(\mathbb{C}) = \mathbb{C}\langle e, \frac{1}{2}e^{-1}h \rangle / ([e, \frac{1}{2}e^{-1}h] - 1)$. Similarly we can establish an isomorphism on the level of Rees algebras

$$R(U_z)[(\hbar^2 e)^{-\frac{1}{2}}] \cong R(W_1(\mathbb{C}))[(\hbar^2 e)^{-\frac{1}{2}}] \otimes_{\mathbb{C}[[\hbar]]} R(\mathcal{O}^{\lambda(z)})$$

(recall that $\hbar^2 e \in R(U_z)$). Since, $R(U_z)_{\bar{a}}$ is complete with respect to $(\hbar^2 e - 1)$, we have that $(\hbar^2 e)^{-\frac{1}{2}} \in R(U_z)_{\bar{a}}$. This and the above isomorphism yields the desired isomorphism $R(U_z)_{\bar{a}} \cong W_{\hbar}(Y_{\bar{a}}) \otimes_{\mathbb{C}[[\hbar]]} R(\mathcal{O}_{\lambda(z)})_{\bar{0}}$.

For generic z , U_z has no finite-dimensional representations (this was shown in [KT]). Thus, simplicity of U_z for generic z follows from Proposition 2.1, since $\mathcal{O}^{\lambda(z)}$ has no nontrivial finite-dimensional representations if $\lambda(z) \cdot \alpha \neq 0$ for all non-Dynkin roots, according to [CBH]. \square

4. ANALOGUE OF BERNSTEIN'S INEQUALITY

We start by recalling few standard results (whose proofs are given for the convenience of the reader), which will be used for the proofs of Theorem 4.1, Theorem 4.2.

The proof of the following Proposition follows directly a well-known proof of Bernstein's inequality given by Joseph [GM].

Proposition 4.1. *Let R be a Noetherian ring with finite GK-dimension and let M be a finitely generated $W_1(R)$ -module. Then for any finite dimensional \mathbb{C} -subspace $N \subset M$*

we have $GK_R(RN) \leq GK_{W_1(R)}M + GK_{W_1(\mathbb{C})}(W_1(\mathbb{C})N) - 2$. In particular $GK_R(RN) \leq 2(GK_{W_1(R)}(M) - 1)$.

Proof. Let R_i be an ascending filtration of R by finite dimensional \mathbb{C} -spaces, such that $R_0 = \mathbb{C}, R_n R_m = R_{n+m}$. Denote by A_n the n -th degree part of the Bernstein filtration on $W_1(\mathbb{C})$, so $A_n = \sum_{i+j \leq n} \mathbb{C}x^i y^j$. Put $U_n = \sum_{i \leq n} A_i R_{n-i}$. Also put $N^n = U_n N$. It is easy to check that the kernel of the

multiplication map $f_n : U_n \rightarrow \text{Hom}_{\mathbb{C}}(A_n N, N^{2n})$ is $\sum A_i(R_{n-i} \cap \text{Ann}(N))$. Thus, $\sum A_i(R_{n-i}/\text{Ann}(N))$ injects into $\text{Hom}_{\mathbb{C}}(A_n N, N^{2n})$. So,

$$\frac{1}{\dim N} \sum_{i \leq n} (i+1) \dim R_{n-i} N \leq \dim N^{2n} \dim A_n N.$$

Therefore

$$GK_R(RN) \leq GK_{W_1(R)}M + GK_{W_1(\mathbb{C})}(W_1(\mathbb{C})N) - 2 \leq 2(GK_{W_1(R)}(M) - 1).$$

□

Corollary 4.1. *Let R be a Noetherian ring with finite GK-dimension and let M be a finitely generated $W_1(R)$ -module such that $GK_{W_1(R)}(M) = 1$. Then $GK(R/\text{Ann}_R(M)) = 0$.*

Proof. Let $N \subset M$, $\dim N < \infty$ be such that $W_1(R)N = M$. Then by Proposition 4.1, $GK_R(RN) = 0$, so $GK(R/\text{Ann}_R(RN)) = GK(R/\text{Ann}_R(M)) = 0$. □

Lemma 4.1. *Let R be an affine Noetherian \mathbb{C} -algebra, and let $e \in R$ be an element such that $\text{ad}(e)$ is locally nilpotent on R . Then for any finitely generated R -module M , we have $GK_{R[e^{-\frac{1}{2}}]}M[e^{-\frac{1}{2}}] \leq GK_R M$.*

Proof. Without loss of generality we may assume that M is e -torsion free. Let $V \subset R$ be an $\text{ad}(e)$ -stable finite dimensional generating subspace of R such that $1, e \in V$. Let $M_0 \subset M$ be a subspace such that $RM_0 = M$, $\dim_{\mathbb{C}} M_0 < \infty$. Since $e^{-i}(eV^{n-i-1}) \subset e^{-(i-1)}V^{n-(i-1)}$, we get that

$$\dim_{\mathbb{C}} \left(\sum_{i \leq n} e^{-i} V^{n-i} M_0 \right) \leq \dim_{\mathbb{C}} \sum_{i \leq n} (V^{n-i} M_0 / e V^{n-i-1} M_0).$$

Therefore, $\dim_{\mathbb{C}} \sum_{i \leq n} e^{-i} V^{n-i} M_0 = O(n^d)$ where $d = GK_R M$. There exists $m > 0$, such that $Ve^{-1} \subset \sum_{i \leq m} e^{-i} V$, $Ve^{\frac{1}{2}} \subset e^{\frac{1}{2}} \sum_{i \leq m} e^{-i} V$. Put $L = \mathbb{C}e^{-1} + \mathbb{C}e^{\frac{1}{2}} + V$. Then L generates $R[e^{-\frac{1}{2}}]$ and $L^n \subset \sum_{i \leq mn} (e^{-i} V^{mn-i} + e^{\frac{1}{2}} e^{-i} V^{mn-i})$. Thus we conclude that $\dim L^n M_0 = O(n^d)$, so $GK_{R[e^{-\frac{1}{2}}]}M[e^{-\frac{1}{2}}] \leq d$. □

Lemma 4.2. *Let M be a finitely generated R -module which can be filtered with R -submodules $M_0 \subset \dots \subset M_n = M$, such that $GK(M_i/M_{i-1}) \geq \frac{1}{2}GK(R/\text{Ann}(M_i/M_{i-1}))$ for all $i = 1, \dots, n$.*

Then $GK(M) \geq \frac{1}{2}GK(R/\text{Ann}(M))$.

Proof. We will proceed by induction on i to show that $GK(M_i) \geq \frac{1}{2}GK(R/\text{Ann}(M_i))$. Put $I = \text{Ann}(M_{i-1})$, $J = \text{Ann}(M_i/M_{i-1})$ then

$$GK(H/\text{Ann}(M_i)) \leq GK(R/IJ) \leq \text{Max}\{GK(R/I), GK(R/J)\}$$

and $GK(M_i) \geq \text{Max}\{GK(M_{i-1}), GK(M_i/M_{i-1})\}$.

Thus $GK(M_i) \geq \frac{1}{2}GK(R/\text{Ann}(M_i))$. □

Theorem 4.1. *For any z and any finitely generated U_z -module M , one has $GK(M) \geq \frac{1}{2}GK(U_z/\text{Ann}(M))$.*

Proof. Recall that $V(I)$ is the singular locus of $\text{Spec gr } U_z = \text{Spec } B_n$, where $I = (x, y, \Delta)$. Let M be a finitely generated U_z -module. If $SS(M)$ (the characteristic variety of M) intersects with $\text{Spec } B_n \setminus V(I)$ (which is the smooth locus of $\text{Spec } B_n$), then it follows from Gabber's theorem that $SS(M) \cap (\text{Spec } B_n \setminus V(I))$ is a coisotropic subvariety of a symplectic variety $\text{Spec } B_n \setminus V(I)$, thus $GK(M) \geq \frac{1}{2}GK(U_z)$. On the other hand, If $SS(M) = \{0\}$, then $\dim M$ is finite, so $GK(U_z/\text{Ann}(M)) = 0$ and we are done. Therefore, we may assume that $\{0\} \neq SS(M) \subset V(I)$.

let $a \in SS(M)$, $a \neq 0$. We will proceed by the induction on the number of irreducible components of $SS(M)$. If $SS(M) = V(I)$, then $GK(M) = 2$ and there is nothing to prove. Thus we may assume that $SS(M)$ is a union of finitely many lines in $V(I)$ through the origin, equivalently $GK(M) = 1$. We may assume without loss of generality that $a = (e - 1, f, h)$. In particular, $e \notin \sqrt{\text{Ann}(\text{gr } M)}$. Denote by $M' = \{m \in M : e^n m = 0, n \gg 0\}$. Thus, M' is a U_z -submodule of M and $M'' = M/M'$ is e -torsion free U_z -module. Then, $SS(M') \subset SS(M)$ and $a \notin SS(M')$. Hence the number of irreducible components of $SS(M')$ is less than that of $SS(M)$. Therefore, by the induction assumption $GK(U_z/\text{Ann}(M')) \leq 2$. By Lemma 4.1, we have that $GK_{U_z[e^{-\frac{1}{2}}]}(M''[e^{-\frac{1}{2}}]) \leq 1$. Let us denote $\mathbb{C}[e^{-1}h, e - 1] \otimes U'_z = W_1(U'_z)$ by U''_z . As was shown in the proof of Theorem 3.1, $e \in W_1(\mathcal{O}^{\lambda(z)})$, $U_z[e^{-1/2}] = W_1(\mathcal{O}^{\lambda(z)})[e^{-1/2}]$. Let $N \subset M''[e^{-\frac{1}{2}}]$ be a finitely generated $W_1(\mathcal{O}^{\lambda(z)})$ -module which generates $M''[e^{-\frac{1}{2}}]$ over $U_z[e^{-\frac{1}{2}}]$. Thus, $N[e^{-\frac{1}{2}}] = M''[e^{-\frac{1}{2}}]$. Then (by Lemma 4.1) $GK_{W_1(\mathcal{O}^{\lambda(z)})}(N) \leq GK_{U_z[e^{-\frac{1}{2}}]}(M''[e^{-\frac{1}{2}}]) \leq 1$. Therefore, $GK(\mathcal{O}^{\lambda(z)}/\text{Ann}(N)) = 0$ by Corollary 4.1. Then, there is a nonzero polynomial g , such that $g(\Delta)N = 0$. Similarly, there is a nonzero polynomial ϕ , such that $\phi(e^{-1}x^2) \in \text{Ann}(N)$ (recall that $\Delta, e^{-1}x^2 \in \mathcal{O}^{\lambda(z)}$). Hence $e^m \phi(e^{-1}x^2) \in \text{Ann}(M'')$ for all m . Choosing m , such that $e^m \phi(e^{-1}x^2) \in U_z$, we conclude that there is a nonzero $\psi \in \mathbb{C}[e, x]$, such that $\psi \in \text{Ann}(N)$. Since $g(\Delta), \psi$ commute with $e^{-\frac{1}{2}}$ and $M'' \subset \sum_i e^{-\frac{i}{2}} N$ we conclude that $g(\Delta), \psi \in \text{Ann}_{U_z} M''$. So, $GK(U_z/\text{Ann}(M'')) \leq 2$. Applying Lemma 4.2 we are done. \square

Theorem 4.2. *If M is a finitely generated H_z -module, then $GK_{H_z}(M) \geq \frac{1}{2}GK(H_z/\text{Ann}(M))$*

Proof. Throughout we will suppress index z for simplicity. Let M be a finitely generated $\mathbb{C}[t]$ -torsion free H -module. Let $\rho : \mathbb{C}(t) \rightarrow F$ be an embedding into an algebraically closed field F . Put $H_F = H \otimes_{\mathbb{C}} F$. Then $U_F = H_F/(t - \rho(t)) = H(t) \otimes_{\mathbb{C}(t)} F$ and $M_F = M(t) \otimes_{\mathbb{C}(t)} F$ is a finitely generated module over U_F . So, we may apply Theorem 4.1 to U_F, M_F (where instead of \mathbb{C} the ground field is F). Thus, $2GK_{U_F}(M_F) \geq GK_F(U_F/\text{Ann}(M_F))$. But $\text{Ann}_{U_F} M_F = \text{Ann}_{H(t)}(M(t) \otimes_{\mathbb{C}(t)} F)$, so

$U_F/Ann(M_F) = (H(t)/AnnM(t)) \otimes_{\mathbb{C}(t)} F$. Therefore, $2GK_{H(t)}(M(t)) \geq GK(H(t)/Ann(M))$. On the other hand $GK_H(M) - 1 = GK_{H(t)}(M(t))$ and $GK_{\mathbb{C}(t)}(H(t)/Ann(M)) = GK(H/Ann(M)) - 1$.

Thus, $GK_H(M) \geq \frac{1}{2}GK(H/Ann(M))$.

Let M be an arbitrary finitely generated H -modules. Let $M' \subset M$ be the submodule of all $\mathbb{C}[t]$ -torsion modules. Thus, M/M' is $\mathbb{C}[t]$ -torsion free and M' can be filtered by H -submodules, such that each subquotient is annihilated by some $t - \lambda$ for some $\lambda \in \mathbb{C}$. Then M can be filtered by H -submodules, such that for each subquotient the conclusion of the theorem holds. By Lemma 4.2 we are done. \square

5. EQUIVALENCE

In this section we establish an equivalence between the category of (generalized) Whittaker U_z -modules and the category of $\mathcal{O}^{\lambda(z)}$ -modules, an analogue of Skryabin's equivalence [S]. This equivalence is a direct consequence of the exactness of the Whittaker functor for \mathfrak{sl}_2 (simplest case of Kostant's result) and the isomorphism $U_z[e^{\frac{-1}{2}}] \cong W_1(\mathcal{O}^{\lambda(z)})$.

Let us briefly recall the setup of the quantum hamiltonian reduction of algebras. Let A be an associative \mathbb{C} -algebra, and let $\mathfrak{n} \subset A$ be a finite dimensional nilpotent Lie subalgebra under the commutator bracket of A . Suppose that $\text{ad}(\mathfrak{n})$ action of \mathfrak{n} on A is locally nilpotent. Let us denote $(A/\text{An})^{\mathfrak{n}}$ by $\mathbb{H}(\mathfrak{n}, A)$. Then $\mathbb{H}(\mathfrak{n}, A) = \text{End}_A(A/\text{An})^{op}$ is an algebra. The full subcategory of A -modules consisting of those A -modules on which \mathfrak{n} acts locally nilpotently (Whittaker modules) will be denoted by $(A, \mathfrak{n})\text{-mod}$. One has a functor Wh (Whittaker functor) from $(A, \mathfrak{n})\text{-mod}$ to the category of $\mathbb{H}(\mathfrak{n}, A)$ -modules defined as follows $Wh_A(M) = M^{\mathfrak{n}} = \text{Hom}_A(A/\text{An}, M)$. There is a functor in the opposite direction $F(N) = A/\text{An} \otimes_{\mathbb{H}(\mathfrak{n}, A)} N$. Under these assumptions we have the following standard

Proposition 5.1. *Let $B \subset A$ be a \mathbb{C} -subalgebra containing \mathfrak{n} . If $H^i(\mathfrak{n}, B/B\mathfrak{n}) = 0$ for all $i > 0$, then the functor Wh_A induces an equivalence. Moreover, if A/An is a faithfully flat right $\mathbb{H}(\mathfrak{n}, A)$ -module, then the inverse is given by the functor F .*

Proof. At first we check that the functor Wh_B is an exact functor. Of course this will imply the same about Wh_A . It suffices to check that for any B -module M on which \mathfrak{n} acts locally nilpotently, one has $H^i(\mathfrak{n}, M) = 0$ for all $i > 0$. Let m be a nonnegative integer. Assume that for any such M one has $H^i(\mathfrak{n}, M) = 0$ for all $i > m$. This is obviously true for $m = \dim \mathfrak{n}$. We will proceed by descending on m . Let C be the full subcategory of all $(B, \mathfrak{n})\text{-mod}$ whose objects are M , such that $H^m(\mathfrak{n}, M) = 0$. Clearly Wh is exact on C . Also, C is closed under taking quotients, arbitrary direct sums, extensions and contains $B/B\mathfrak{n}$. Let N be an object in (B, \mathfrak{n}) . Let N' be the sum of all submodules of N that belong to C . Then N' belongs to C and

no nontrivial submodule of N/N' belongs to C . This implies that $N/N' = 0$, otherwise it will contain a nonzero quotient of $B/B\mathfrak{n}$, a contradiction. Thus, N belongs to C , and Wh_A is an exact functor. This implies that $A/A\mathfrak{n}$ is a projective generator of $(A, \mathfrak{n})\text{-mod}$ since $Wh_A = Hom_A(A/A\mathfrak{n}, -)$. Therefore, $(A, \mathfrak{n})\text{-mod}$ is equivalent to $End_A(A/A\mathfrak{n})^{op} = \mathbb{H}(\mathfrak{n}, A)$ with Wh_A being an equivalence.

Now suppose that $A/A\mathfrak{n}$ is a flat right $\mathbb{H}(\mathfrak{n}, A)$ -module. Then, F is an exact functor, and so is $Wh_A(F) : \mathbb{H}(\mathfrak{n}, A)\text{-mod} \rightarrow \mathbb{H}(\mathfrak{n}, A)\text{-mod}$. Clearly Id is a subfunctor of $Wh(F)$, moreover $Wh(F(\mathbb{H}(\mathfrak{n}, A))) = \mathbb{H}(\mathfrak{n}, A)$. Therefore, since $Wh(F)$ preserves direct sums, we get that $Wh(F) = Id$. \square

Recall that \mathcal{O}^λ denoted the noncommutative deformation of Kleinian singularity of type D_{n+2} with parameter λ .

Theorem 5.1. *Let c be a nonzero complex number. One has $\mathcal{O}^{\lambda(z)} = (U_z/U_z(e-c))^{e-c}$. The functor $M \rightarrow Wh(M) = \{m \in M : (e-c)m = 0\}$ defines an equivalence between the category of U_z -modules on which $(e-c)$ acts locally nilpotently and the category of $\mathcal{O}^{\lambda(z)}$ -modules, the inverse functor is given by $N \rightarrow F(N) = U_z/U_z(e-c) \otimes_{\mathcal{O}^{\lambda(z)}} N$.*

Proof. Without loss of generality assume that $c = 1$. It is well-known and easy to check that $H^1(\mathbb{C}e, \mathfrak{sl}_2/\mathfrak{sl}_2(e-1)) = 0$. Thus, according to Proposition 5.1, it suffices to check that $\mathcal{O}^{\lambda(z)}$ is isomorphic to $(U_z/U_z(e-1))^{e-1}$ and $U_z/U_z(e-1)$ is a free right $(U_z/U_z(e-1))^{e-1}$ -module.

It was shown in the proof of Theorem 3.1 that $U_z[e^{-\frac{1}{2}}] = A[e^{-\frac{1}{2}}]$, where $A = W_1(\mathcal{O}^{\lambda(z)})[e^{-\frac{1}{2}}]$. Thus, $U_z/U_z(e-1) = A/A(e^{-\frac{1}{2}} - 1)$, and $(U_z/U_z(e-1))^e = (A/A(e^{-\frac{1}{2}} - 1))^e$. But $A/A(e^{-\frac{1}{2}} - 1) = \mathbb{C}[e^{-1}h] \otimes \mathcal{O}^{\lambda(z)}$ and $(A/A(e^{-\frac{1}{2}} - 1))^e = \mathcal{O}^{\lambda(z)}$ and we are done. \square

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THE UNIVERSITY OF TOLEDO, DEPARTMENT OF MATHEMATICS, TOLEDO, OHIO, USA
E-mail address: `tikar06@gmail.com`